Lecture 18. Linear transformations of vector spaces

Def A function $T: V \longrightarrow W$ between two vector spaces V and W is called a <u>linear transformation</u> if it has the following properties:

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v} \in V$

(ii) $T(c\vec{v}) = cT(\vec{v})$ for any $c \in \mathbb{R}$, $\vec{v} \in V$

Note (1) We can use bases of V and W to express T by a matrix multiplication.

e.g. $V = \mathbb{P}_2$, $W = \mathbb{R}^3$ with standard bases

$\Longrightarrow T: \mathbb{P}_2 \longrightarrow \mathbb{R}^3$ has standard matrix with columns $T(1), T(t), T(t^2)$.

(2) Once T is written as a multiplication by a matrix A, we can use RREF(A) to study various properties of T as follows:

- T is injective $\Longleftrightarrow$ RREF(A) has a leading 1 in every column
- T is surjective $\Longleftrightarrow$ RREF(A) has a leading 1 in every row
- T is invertible $\Longleftrightarrow$ RREF(A) $= I$

(3) In Math 313, we will mostly consider linear transformations on $\mathbb{R}^n$ or $\mathbb{P}_n$.

(4) We have $T(\vec{0}) = \vec{0}$

Ex Determine whether each function is a linear transformation.

(1) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{P}_2$ given by

$$T_1(\vec{x}) = (x_1 - 2x_2) + 3x_1 t + 2x_2 t^2$$

Sol $$[T_1(\vec{x})] = \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 \\ 2x_2 \end{bmatrix}$$ linear coordinates with no constant terms

Hence T_1 is $\boxed{\text{a linear transformation}}$

(2) $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{P}_2$ given by

$$T_2(\vec{x}) = (x_1^2 - 2x_2) + 3x_1 t + 2x_2^3 t^2$$

Sol $$[T_2(\vec{x})] = \begin{bmatrix} x_1^2 - 2x_2 \\ 3x_1 \\ 2x_2^3 \end{bmatrix}$$ nonlinear terms in coordinates

Hence T_2 is $\boxed{\text{not a linear transformation}}$

(3) $T_3 : \mathbb{P}_3 \longrightarrow \mathbb{R}^2$ given by

$$T_3(p(t)) = \begin{bmatrix} p(2) \\ p(0) \end{bmatrix}.$$

Sol For polynomials $p(t), q(t) \in \mathbb{P}_3$, we have

$$T_3(p(t)+q(t)) = \begin{bmatrix} p(2)+q(2) \\ p(0)+q(0) \end{bmatrix} = \begin{bmatrix} p(2) \\ p(0) \end{bmatrix} + \begin{bmatrix} q(2) \\ q(0) \end{bmatrix} = T_3(p(t)) + T_3(q(t))$$

For a polynomial $p(t) \in \mathbb{P}_3$ and a scalar $c \in \mathbb{R}$, we have

$$T_3(cp(t)) = \begin{bmatrix} cp(2) \\ cp(0) \end{bmatrix} = c\begin{bmatrix} p(2) \\ p(0) \end{bmatrix} = cT_3(p(t))$$

Hence T_3 is $\boxed{\text{a linear transformation}}$

(4) $T_4 : \mathbb{P}_2 \longrightarrow \mathbb{P}_5$ given by

$$T_4(p(t)) = (t^3-2)\,p(t).$$

Sol For polynomials $p(t), q(t) \in \mathbb{P}_2$, we have

$$\begin{aligned} T_4(p(t)+q(t)) &= (t^3-2)(p(t)+q(t)) \\ &= (t^3-2)\,p(t) + (t^3-2)\,q(t) = T_4(p(t)) + T_4(q(t)) \end{aligned}$$

For a polynomial $p(t) \in \mathbb{P}_2$ and a scalar $c \in \mathbb{R}$, we have

$$T_4(cp(t)) = (t^3-2)\cdot cp(t) = c(t^3-2)\,p(t) = c\,T_4(p(t))$$

Hence T_4 is $\boxed{\text{a linear transformation}}$

(5) $T_5 : \mathbb{P}_2 \longrightarrow \mathbb{P}_5$ given by

$$T_5(p(t)) = t\,p(t)^2.$$

Sol $T_5(t) = t\cdot t^2 = t^3$ and $T_5(2t) = t\cdot(2t)^2 = 4t^3 \Longrightarrow T_5(2t) \neq 2T_5(t)$

Hence T_5 is $\boxed{\text{not a linear transformation}}$

Note The main issue is the nonlinear term $p(t)^2$ for the input $p(t)$.

(6) $T_6 : \mathbb{P}_4 \longrightarrow \mathbb{P}_3$ given by

$$T_6(p(t)) = 3p'(t) + 2p''(t)$$

Sol For polynomials $p(t), q(t) \in \mathbb{P}_4$, we have

$$\begin{aligned} T_6(p(t)+q(t)) &= 3(p(t)+q(t))' + 2(p(t)+q(t))'' \\ &= 3p'(t) + 3q'(t) + 2p''(t) + 2q''(t) \\ &= (3p'(t)+2p''(t)) + (3q'(t)+2q''(t)) = T_6(p(t)) + T_6(q(t)) \end{aligned}$$

For a polynomial $p(t) \in \mathbb{P}_2$ and a scalar $c \in \mathbb{R}$, we have

$$\begin{aligned} T_6(cp(t)) &= 3(cp(t))' + 2(cp(t))'' = 3cp'(t) + 2cp''(t) \\ &= c(3p'(t)+2p''(t)) = c\,T_6(p(t)) \end{aligned}$$

Hence T_6 is $\boxed{\text{a linear transformation}}$

Ex Consider the linear transformation $T : \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ given by

$$T(p(t)) = \begin{bmatrix} p(1) \\ p'(2) \end{bmatrix}.$$

(1) Find the standard matrix A of T.

Sol A has columns $T(1), T(t), T(t^2)$.

$$p(t)=1 : p(1)=1, \ \underbrace{p'(2)=0}_{p'(t)=(1)'=0} \Rightarrow T(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$p(t)=t : p(1)=1, \ \underbrace{p'(2)=1}_{p'(t)=(t)'=1} \Rightarrow T(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$p(t)=t^2 : p(1)=1, \ \underbrace{p'(2)=4}_{p'(t)=(t^2)'=2t} \Rightarrow T(t^2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Hence the standard matrix is $A = \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}}$

(2) Determine whether T is injective

Sol $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix}$ has no leading 1's in column 3

$\Rightarrow T$ is $\boxed{\text{not injective}}$

(3) Determine whether T is surjective

Sol $\text{RREF}(A)$ has a leading 1 in every row

$\Rightarrow T$ is $\boxed{\text{surjective}}$